New Non-traveling Solitary Wave Solutions for a Second-order Korteweg-de Vries Equation

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Modeling the propagation of two different wave modes simultaneously, the second-order KdV equation is of current interest. Applying a tanh-typed method with symbolic computation, we have found certain new analytic soliton-typed solutions which go beyond the the previously obtained traveling wave solutions.

Key words: Nonlinear Evolution Equations; Second-order KdV Equation; Solitonic Solutions; Symbolic Computation.

We investigate the second-order KdV equation proposed by Korsunsky [1], which is assumed to govern propagation in the same direction of two wave modes with the same dispersion relation, but with different phase velocities, nonlinearity and dispersion parameters:

$$u_{xx} + (c_1 + c_2)u_{xt} + c_1 c_2 u_{xx} + \left[(\alpha_1 + \alpha_2)\frac{\partial}{\partial t} + (\alpha_1 c_2 + \alpha_2 c_1)\frac{\partial}{\partial x}\right] u u_x + \left[(\beta_1 + \beta_2)\frac{\partial}{\partial t} + (\beta_1 c_2 + \beta_2 c_1)\frac{\partial}{\partial x}\right] u u_{xxxx} = 0, \quad (1)$$

where $u(x,t)$ is a field function, c_i are the phase velocities, α_i the parameters of nonlinearity, and β_i the dispersion parameters for the first ($i = 1$) and second ($i = 2$) mode. This equation exhibits two important features: (i) if one of the modes is absent, the other obeys the ordinary KdV equation, and (ii) on application of the perturbation techniqu,e this equation leads to the uncoupled KdV equations for each mode on a corresponding temporal and spatial scale [1]. We can show that in the absence of the other wave the evolution of each mode is described by its own KdV equation

$$u_t + c_i u_x + \alpha_i u u_x + \beta_i u_{xxx} = 0 \quad (2)$$

with the traveling solitary wave solutions

$$u(x,t) = A \cdot \mathrm{sech}^2\{[x - (c_i + \frac{1}{3}\alpha_1 A)t]/L_i\}, \quad (3)$$
where $12L_i^2 = A\alpha_i/\beta_i$.

To simplify the analysis, we transform (1), using the transformations

$$\chi = (\beta_1 + \beta_2)^{-1/2}(x - c_0 t),\ T = (\beta_1 + \beta_2)^{-1/2} t, \quad c_0 = \frac{1}{2}(c_1 + c_2),\ U(\chi, T) = (\alpha_1 + \alpha_2)u(x,t), \quad (4)$$

into

$$U_{TT} - s^2 U_{\chi\chi} + \left(\frac{\partial}{\partial T} + \alpha s\frac{\partial}{\partial \chi}\right) U U_\chi + \left(\frac{\partial}{\partial T} + \beta s\frac{\partial}{\partial \chi}\right) U U_{\chi\chi\chi} = 0, \quad (5)$$

where

$$s = \frac{1}{2}(c_1 - c_2),\quad \alpha = \frac{\alpha_2 - \alpha_1}{\alpha_2 + \alpha_1},\quad \beta = \frac{\beta_2 - \beta_1}{\beta_2 + \beta_1} \quad (6)$$

with $s > 0$, $|\alpha| \le 1$, and $|\beta| \le 1$.

Two families of traveling wave solutions have been found in [1] for (5):

$$U^{\mathrm{I}}(x,t) = U_0 + 3U_0 \cdot \mathrm{sech}^2\Big[\pm \sqrt{\frac{U_0(\alpha \pm 1)}{4(\beta \pm 1)}} \Big((\beta_1 + \beta_2)^{-1/2}(x - c_0 t) - \alpha t\Big) \Big], \tag{7}$$

where U_0 is a constant wave amplitude and $\alpha = \pm s$ corresponds to the two modes represented in (1) and

$$U^{\mathrm{II}}(x,t) = A \cdot \mathrm{sech}^2\Big[\pm \sqrt{\frac{A(\alpha s - \alpha)}{12(\beta s - \alpha)}} \Big((\beta_1 + \beta_2)^{-1/2}(x - c_0 t) - \alpha t\Big) \Big], \tag{8}$$

where the two roots of $\alpha^2 = s^2 + 1/3A(\alpha s - \alpha)$ correspond to the two solitary wave modes, and A is the wave amplitude.

In this work we apply the tanh-typed method [2 - 4] with symbolic computation to (5) to find some non-traveling solitary wave solutions. As an Ansatz we assume that the physical field $U(\chi, T)$ has the form

$$U(\chi,T) = \sum_{n=0}^{N} A_n(T) \cdot \tanh^n\left[G(T) \cdot \chi + H(T)\right], \tag{9}$$

where N is the integer determined via the balance of the highest-order contributions from both the linear and nonlinear terms of (5) as $N = 2$, while $A_N(t)$, $G(T)$, and $H(T)$ are the non-trivial differentiable functions to be determined.

With the symbolic computation package *Maple* we substitute the Ansatz (9), together with the above conditions, into (5) and collect the coefficients of like powers of tanh:

$$\begin{aligned}(\tanh^6):\ & 10\,A_2(T)G(T)[A_2(T)\alpha\, sG(T) + A_2(T)G(T)_T \cdot \chi + A_2(T)H(T)_T + 12\,\beta\, sG(T)^3 \\ & +\ 12G(T)^2G(T)_T\chi + 12\,G(T)^2H(T)_T]\end{aligned} \tag{10}$$

$$\begin{aligned}(\tanh^5):\ & -2\,A_2(T)^2G(T)_T - 24\,A_2(T)_{TT}G(T)^3 + 12\,A_1(T)G(T)_TA_2(T)G(T)\cdot\chi \\ & +\ 24A_1(T)G(T)^3G(T)_T\cdot\chi + 24\beta sA_1(T)G(T)^4 - 72A_2(T)G(T)^2G(T)_T + 24A_1(T)G(T)^3H(T)_T \\ & +\ 12\,\alpha\, sA_1(T)G(T)^2A_2(T) + 12\,A_1(T)H(T)_TA_2(T)G(T) - 4\,A_2(T)_{TT}A_2(T)G(T)\end{aligned} \tag{11}$$

$$\begin{aligned}(\tanh^4):\ & -6\,A_1(T)_T(G(T))^3 + 6\,A_2(T)H(T)_T{}^2 - 6\,s^2A_2(T)(G(T))^2 + 3A_1(T)^2G(T)_TG(T)\cdot\chi \\ & -\ 3\,A_1(T)_TA_2(T)G(T) + 6\,A_0(T)A_2(T)G(T)H(T)_T - 240\,A_2(T)(G(T))^3H(T)_T \\ & -\ 240\,A_2(T)G(T)^3G(T)_T\cdot\chi - 18\,A_1(T)G(T)^2G(T)_T + 6\,A_2(T)G(T)_T{}^2\cdot\chi^2 \\ & +\ 6\,A_0(T)A_2(T)G(T)G(T)_T\cdot\chi - 240\,\beta\, sA_2(T)G(T)^4 - 16A_2(T)^2H(T)_TG(T) \\ & +\ 12\,A_2(T)G(T)_TH(T)_T\cdot\chi - 3\,A_2(T)_{TT}A_1(T)G(T) - 16\,\alpha\, s(A_2(T))^2(G(T))^2 \\ & +\ 3\,A_1(T)^2H(T)_TG(T) + 3\,\alpha\, s(A_1(T))^2G(T)^2 - 16\,(A_2(T))^2G(T)_TG(T)\cdot\chi \\ & -\ 3\,A_1(T)A_2(T)G(T)_T + 6\,\alpha\, sA_0(T)A_2(T)(G(T))^2\end{aligned} \tag{12}$$

$$\begin{aligned}(\tanh^3):\ & -18\,A_1(T)H(T)_TA_2(T)G(T) - 40A_1(T)G(T)^3H(T)_T - 40\,A_1(T)G(T)^3G(T)_T\cdot\chi \\ & -\ 18\,A_1(T)G(T)_TA_2(T)G(T)\cdot\chi - 4\,A_2(T)_TH(T)_T + 120\,A_2(T)G(T)^2G(T)_T \\ & -\ 2\,A_2(T)G(T)_{TT}\cdot\chi - A_1(T)^2G(T)_T - 2\,A_2(T)H(T)_{TT} + 2\,A_1(T)H(T)_T{}^2 \\ & +\ 2\,A_1(T)G(T)_T{}^2\cdot\chi^2 - 2\,A_0(T)A_2(T)G(T)_T - 4\,A_2(T)_TG(T)_T\cdot\chi - 2\,A_1(T)_TA_1(T)G(T)\end{aligned} \tag{13}$$

$$+\ 4\,A_2(T)_T A_2(T)G(T) - 2\,A_0(T)A_2(T)_T G(T) - 2\,A_0(T)_T A_2(T)G(T) + 40\,A_2(T)_T G(T)^3$$
$$-\ 40\,\beta\, sA_1(T)G(T)^4 - 18\,\alpha\, sA_1(T)G(T)^2 A_2(T) - 2\,s^2 A_1(T)G(T)^2$$
$$+\ 2\,A_0(T)A_1(T)G(T)G(T)_T\cdot\chi + 2\,\alpha\, sA_0(T)A_1(T)G(T)^2 + 2\,A_0(T)A_1(T)G(T)H(T)_T$$
$$+\ 4\,A_1(T)G(T)_T H(T)_T\cdot\chi + 2A_2(T)^2 G(T)_T$$

$$(\tanh^2): A_2(T)_{TT} - 4A_1(T)^2 H(T)_T G(T) + 6A_2(T)^2 H(T)_T G(T) + 136\,A_2(T)G(T)^3 H(T)_T \quad (14)$$
$$+\ 3\,A_1(T)A_2(T)G(T)_T - 8\,A_2(T)G(T)_T{}^2\cdot\chi^2 + 24A_1(T)G(T)^2 G(T)_T - 4A_1(T)^2 G(T)_T G(T)\cdot\chi$$
$$+\ 3\,A_1(T)_T A_2(T)G(T) - 16\,A_2(T)G(T)_T H(T)_T\cdot\chi - 8\,A_0(T)A_2(T)G(T)H(T)_T$$
$$-\ 8A_0(T)A_2(T)G(T)G(T)_T\cdot\chi + 136\,A_2(T)G(T)^3 G(T)_T\cdot\chi + 136\beta\, sA_2(T)G(T)^4$$
$$-\ 4\,\alpha\, sA_1(T)^2 G(T)^2 - 8\,\alpha\, sA_0(T)A_2(T)G(T)^2 + 6A_2(T)^2 G(T)_T G(T)\cdot\chi + 8\,s^2 A_2(T)G(T)^2$$
$$-\ A_1(T)G(T)_{TT}\cdot\chi - A_0(T)A_1(T)G(T)_T + 6\,\alpha s A_2(T)^2 G(T)^2 + 8\,A_1(T)_T G(T)^3$$
$$-\ 8\,A_2(T)H(T)_T{}^2 - A_0(T)_T A_1(T)G(T) + 3\,A_2(T)_T A_1(T)G(T) - 2\,A_1(T)_T G(T)_T\cdot\chi$$
$$-\ A_0(T)A_1(T)_T G(T) - 2\,A_1(T)_T H(T)_T - A_1(T)H(T)_{TT}$$

$$(\tanh^1): A_1(T)_{TT} + 6\,A_1(T)H(T)_T A_2(T)G(T) + 16\,A_1(T)G(T)^3 H(T)_T + 16\,A_1(T)G(T)^3 G(T)_T\cdot\chi \quad (15)$$
$$+\ 6\,A_1(T)G(T)_T A_2(T)G(T)\cdot\chi + 4\,A_2(T)_T H(T)_T - 48\,A_2(T)G(T)^2 G(T)_T + 2\,A_2(T)G(T)_{TT}\cdot\chi$$
$$+\ A_1(T)^2 G(T)_T + 2\,A_2(T)H(T)_{TT} - 2\,A_1(T)H(T)_T{}^2 - 2\,A_1(T)G(T)_T{}^2\cdot\chi^2$$
$$+\ 2\,A_0(T)A_2(T)G(T)_T + 4\,A_2(T)_T G(T)_T\cdot\chi + 2\,A_1(T)_T A_1(T)G(T) + 2\,A_0(T)A_2(T)_T G(T)$$
$$+\ 2A_0(T)_T A_2(T)G(T) - 16\,A_2(T)_T G(T)^3 + 16\beta s A_1(T)G(T)^4 + 6\,\alpha\, sA_1(T)G(T)^2 A_2(T)$$
$$+\ 2\,s^2 A_1(T)G(T)^2 - 2\,A_0(T)A_1(T)G(T)G(T)_T\cdot\chi - 2\,\alpha\, sA_0(T)A_1(T)G(T)^2$$
$$-\ 2A_0(T)A_1(T)G(T)H(T)_T - 4\,A_1(T)G(T)_T H(T)_T\cdot\chi$$

$$(\tanh^0): A_0(T)_{TT} - 2\,A_1(T)_T G(T)^3 + 2\,A_1(T)_T H(T)_T + 2\,A_2(T)H(T)_T{}^2 + A_1(T)H(T)_{TT} \quad (16)$$
$$+\ A_0(T)A_1(T)G(T)_T + 2\,A_1(T)_T G(T)_T\cdot\chi + 2\,A_2(T)G(T)_T{}^2\cdot\chi^2 - 2s^2 A_2(T)G(T)^2$$
$$+\ A_1(T)^2 H(T)_T G(T) + A_0(T)_T A_1(T)G(T) + A_0(T)A_1(T)_T G(T) + A_1(T)G(T)_{TT}\cdot\chi$$
$$-\ 16\,A_2(T)G(T)^3 H(T)_T - 6\,A_1(T)G(T)^2 G(T)_T + 4A_2(T)G(T)_T H(T)_T\cdot\chi$$
$$-\ 16\,\beta\, sA_2(T)G(T)^4 + \alpha s A_1(T)^2 G(T)^2 + 2\,\alpha\, sA_0(T)A_2(T)G(T)^2 + A_1(T)^2 G(T)_T G(T)\cdot\chi$$
$$+\ 2\,A_0(T)A_2(T)G(T)G(T)_T\cdot\chi + 2\,A_0(T)A_2(T)G(T)H(T)_T - 16\,A_2(T)G(T)^3 G(T)_T\cdot\chi,$$

where the subscript T denotes time derivative.

Our goal is to find the conditions for $A_N(T), G(T)$, and $H(T)$ which simultaneously let the above terms become zero. After dealing with some complicated symbolic calculations using *Maple*, we obtained a new family of non-traveling solitary-wave solutions as

$$U^{\text{new}}(\chi,T) = A_0(T) + A_1(T)\cdot\tanh^1[G(T)\cdot\chi + H(T)] + A_2(T)\cdot\tanh^2[G(T)\cdot\chi + H(T)], \quad (17)$$

where

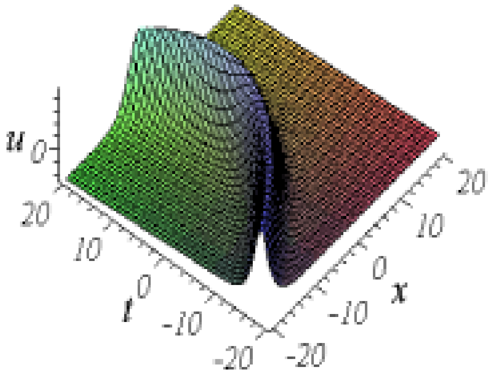

Fig. 1. Beyond traveling solitary-wave solution $U^{\text{new}}(x,t)$ with the parameters $\beta_2 = 10, \beta_1 = 0.01, \alpha_2 = 10, \alpha_1 = 0.01$, $c_1 = 1, c_2 = 0.01, C_1 = 0.1, C_2 = 0.01$, and $C_3 = 0.01$, satisfying the solitary wave property that $U^{\text{new}}(x,t)$ tends to zero $|x|$ as approaches infinity.

$$G(T) = G = \text{nonzero constant}, \tag{18}$$

$$H(T) = C_1T^2 + C_2T + C_3, \tag{19}$$

where C_i are arbitary constants,

$$A_2(T) = -12G^2, \tag{20}$$

$$A_1(T) = 0, \tag{21}$$

$$A_0(T) \equiv \frac{\mathcal{R}(T)}{\mathcal{S}(T)}, \tag{22}$$

$$\begin{aligned}\mathcal{R}(T) \equiv\ & -24\,G(2\,C_1T + C_2)^4 + (192G^4 - 48\,G^2s) \\ & \cdot (2\,C_1T + C_2)^3 + 576\,G^5s(2\,C_1T + C_2)^2 \\ & + (576\,s^2G^6 + 48s^3G^4)(2\,C_1T + C_2) \\ & + 24\,s^4G^5 + 192s^3G^7,\end{aligned}$$

$$\begin{aligned}\mathcal{S}(T) \equiv\ & 3\,G^3s(2\,C_1T + C_2)^2 + 3s^2G^4(2C_1T + C_2) \\ & \cdot G^2(2C_1T + C_2)^3 + s^3G^3,\end{aligned}$$

and the following auxiliary conditions are required for $U^{\text{new}}(\chi, T)$ to be a solution of (5):

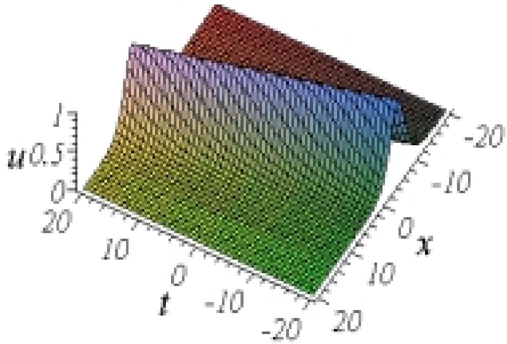

Fig. 2. A typical sech^2-typed solitary wave solution $U^{\text{II}}(x,t)$ with $A = 1, \beta_2 = 10, \beta_1 = 0.01, \alpha_2 = 10, \alpha_1 = 0.01, c_1 = 1$, and $c_2 = 0.01$.

$$\alpha = \beta = 1 \text{ or } \alpha = \beta = -1, \tag{23}$$

which imply that $\alpha_2 \gg \alpha_1$ and $\beta_2 \gg \beta_1$ for $\alpha = \beta = 1$ or $\alpha_1 \gg \alpha_2$ and $\beta_1 \gg \beta_2$ for $\alpha = \beta = -1$. Physically, these conditions indicate that two solitary wave modes propagate in the medium, where one mode's nonlinearity and dispersion parameters are much bigger than the other one's.

Finally we present two figures with some selected parameters. We set $\beta_2 = 10, \beta_1 = 0.01, \alpha_2 = 10, \alpha_1 = 0.01, c_1 = 1, c_2 = 0.01, C_1 = 0.1, C_2 = 0.01$, and $C_3 = 0.01$ for the new solitary-wave solutions (17) and plot $U^{\text{new}}(x,t)$ in Figure 1. The new solutions satisfy solitary wave property that $U^{\text{new}}(x,t)$ tends to zero as $|x|$ approaches infinity. For comparison in Fig. 2 we plot the traveling wave solution of $U^{II}(x,t)$ (8) with $A = 1, \beta_2 = 10, \beta_1 = 0.01, \alpha_2 = 10, \alpha_1 = 0.01, c_1 = 1$, and $c_2 = 0.01$.

To sum up, the tanh method and symbolic computations lead to the new analytic solitary-wave solutions (17), different from the previously obtained results [1] for the second order KdV equation.

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